

Four-body Central Configurations with Adjacent Equal Masses

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Abstract

For any convex non-collinear central configuration of the planar Newtonian 4-body problem with adjacent equal masses $m_1 = m_2 \neq m_3 = m_4$, with equal lengths for the two diagonals, we prove it must possess a symmetry and must be an isosceles trapezoid; furthermore, which is also an isosceles trapezoid when the length between m_1 and m_4 equals the length between m_2 and m_3 .

Keywords: Four-body problem, Convex central configurations, Isosceles trapezoid, Triangle areas

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1. Introduction

It is well known[12][13] that the central configurations for Newtonian n-body problems play an important role in the celestial mechanics. Here, we only consider the planar Newtonian 4-body problem.

The planar Newtonian 4-body problem is related with the motion of 4 point particles with positive masses $m_i \in \mathbb{R}$ and position vectors $q_i \in \mathbb{R}^2$ for $i = 1, \dots, 4$, moving according to Newton's second law and the universal gravitational law:

$$m_i \ddot{q}_i = \frac{\partial U(q)}{\partial q_i}, \quad i = 1, \dots, 4, \quad (1)$$

where

$$U(q) = G \sum_{i < j}^4 \frac{m_i m_j}{r_{ij}} \quad (2)$$

is the Newtonian potential for the 4-body and $r_{ij} = \|q_i - q_j\|$, in the following, we let $G = 1$.

Let $q = (q_1, \dots, q_4) \in (\mathbb{R}^2)^4$ and M be the diagonal mass matrix $\text{diag}(m_1, m_1, \dots, m_4, m_4)$, then the system (1) can be rewritten as the following:

$$\ddot{q} = M^{-1} \frac{\partial U(q)}{\partial(q)}. \quad (3)$$

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To study this problem, without loss of generality, we assume the center of mass is fixed at the origin and consider the space

$$\Omega = \left\{ q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4 \mid \sum_{i=1}^4 m_i q_i = 0 \right\}.$$

Let $\Delta = \bigcup_{i \neq j} \{q \mid q_i = q_j\}$ be the **collision set**. The set $\Omega \setminus \Delta$ is called the **configuration space**.

Here we recall the definition of the central configuration:

Definition. [12][13] A configuration $q \in \Omega \setminus \Delta$ is called a central configuration if there is some constant λ such that

$$M^{-1} \frac{\partial U}{\partial q} = \lambda q. \quad (4)$$

This equation is invariant under rotation, dilatation and reflection on the plane. Two central configurations are considered equivalent if they are related by those symmetry operations.

In 1995 and 1996, Albouy [1][2] proved that there are exactly four equivalent classes for the central configurations of the planar Newtonian 4-body problem with positive equal masses. In 2002, Long and Sun [8] showed that any convex non-collinear central configurations of the planar 4-body problem with equal opposite masses $\beta > \alpha > 0$, such that the diagonal corresponding to the mass α is not shorter than that corresponding to the mass β , must possess a symmetry and must be a kite. Furthermore, it must be a rhombus. In 2003, Albouy [4] showed that any convex non-collinear central configurations of the planar 4-body problem can not be a kite when two pairs of opposite masses are not equal. In 2007, Perez-Chavela and Santoprete [10] generalized the result of Long and Sun[8] and obtained the symmetry of central configurations with equal masses located at opposite vertices of a quadrilateral, but they assume that the two equal masses are not the smallest in all masses. In 2008, Abouy, Fu and Sun [3] proved that, in the planar 4-body problem, a convex central configuration is symmetric with respect to one diagonal if and only if the masses of the two particles on the other diagonal are equal. They also showed that the less massive one is closer to the former diagonal. In this paper, they raised a question: Does the equality of two pairs of adjacent masses implies the configuration is an isosceles trapezoid for co-planar 4-body convex central configurations?

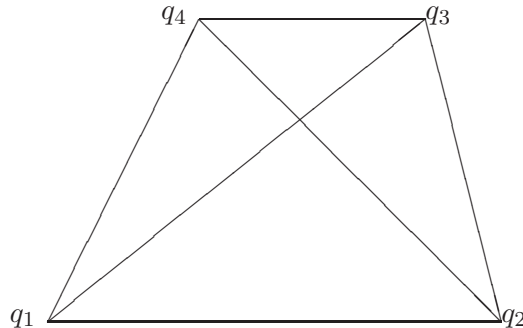


Figure.1

To solve this problem, in 2012, Cors and Roberts [6] used mutual distances as coordinates to study the four-body co-circular central configurations. They had proved that the set of positions that yield co-circular central configurations with positive masses is a two-dimensional surface, the graph of a differentiable function over two of the exterior side-lengths. The boundary of this surface correspond to three important symmetric cases: a kite, an isosceles trapezoid and a degenerate case where three bodies lie at the vertices of an equilateral triangle and the fourth body of the quadrilateral has zero mass; furthermore, they got a stronger result, they only assume that one pair of adjacent masses are equal, then the configuration is an isosceles trapezoid for 4-body convex co-circular central configurations. In 2014, Corbera and Llibre [5] showed that there is a unique convex planar central configuration which has two pairs of equal masses located at the adjacent vertices of the configuration, and which is an isosceles trapezoid when one pair of adjacent masses is sufficiently small.

In this paper, we study the isosceles trapezoid central configurations for the four-body problems. Our main results are:

Theorem 1.1. *Let $q = (q_1, q_2, q_3, q_4) \in \Omega$ be a convex non-collinear central configuration with masses $(\beta, \beta, \alpha, \alpha)$, $\beta > \alpha > 0$. Suppose that the equal masses are at adjacent vertices. If*

$$r_{13} = r_{24},$$

then the configuration q must possess a symmetry, and then forms an isosceles trapezoid.

Theorem 1.2. *Let $q = (q_1, q_2, q_3, q_4) \in \Omega$ be a convex non-collinear central configuration with masses $(\beta, \beta, \alpha, \alpha)$, $\beta > \alpha > 0$. Suppose that the equal masses are at adjacent vertices. If*

$$r_{14} = r_{23},$$

then the configuration q must possess a symmetry, and then forms an isosceles trapezoid.

The arrangement of this paper is as follows. In section 2, we establish our equations for 4-body central configurations by using areas; In section 3, we will prove Theorem 1.2; In section 4, we will give the proof of Theorem 1.3.

2. The central configurations equations by areas of triangles

Firstly, we observe that if $q = (q_1, q_2, q_3, q_4) \in \Omega$ is a central configuration with parameter λ and positive masses (m_1, m_2, m_3, m_4) , then every $\zeta^{-\frac{1}{3}}(q_1, q_2, q_3, q_4) \in \Omega$ is in the same class of central configurations with masses $\zeta^{-1}(m_1, m_2, m_3, m_4)$ and the same value of λ . So, without loss of generality, we suppose $\beta = 1$, and we consider the planar 4-body problem with masses

$$m_1 = m_2 = 1, \quad m_3 = m_4 = \alpha.$$

In this paper, we use Dziobek coordinates, which will be described below. Let

$$a = r_{12}^2, \quad b = r_{13}^2, \quad c = r_{14}^2, \quad d = r_{23}^2, \quad e = r_{24}^2, \quad f = r_{34}^2.$$

For $1 \leq i \leq 4$, let $|\triangle_i|$ be the area of the sub-triangle formed by the remaining three vertices of the configuration q when deleting the point q_i . Then we define the oriented areas of these sub-triangles of the convex non-collinear configuration q by

$$\triangle_1 = -|\triangle_1|, \quad \triangle_2 = |\triangle_2|, \quad \triangle_3 = -|\triangle_3|, \quad \triangle_4 = |\triangle_4|. \quad (5)$$

The above Δ_i satisfy the following equality:

$$\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 0. \quad (6)$$

It is well known[7][11] that the Cayley-Menger determinant

$$S = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & b & c \\ 1 & a & 0 & d & e \\ 1 & b & d & 0 & f \\ 1 & c & e & f & 0 \end{vmatrix}$$

satisfies $S = 0$. In 1900, Dziobek [7](also refer to[6],[9]) proved that

$$\frac{\partial S}{\partial r_{ij}^2} = -32\Delta_i\Delta_j, \quad \text{for all } i \neq j. \quad (7)$$

Let $\varphi(s) = s^{-\frac{1}{2}}$ for $s > 0$. Then the potential function and the momentum of inertia are given by

$$U(q) = \sum_{1 \leq i < j \leq 4} m_i m_j \varphi(r_{ij}^2) \quad (8)$$

and

$$I(q) = \frac{1}{m'} \sum_{1 \leq i < j \leq 4} m_i m_j r_{ij}^2 \quad (9)$$

respectively, where $m' = \sum_{i=1}^4 m_i$.

Using Lagrangian Multiplier Method, Dziobek gave an equivalent characterization of central configurations, they are extremal of

$$U + \lambda S - \mu m'(I - I_0)$$

as a function of $\lambda, \mu m', r_{12}, \dots, r_{34}$, where λ and $\mu m'$ are Lagrange multipliers and I_0 is a fixed moment of inertia. Thus, for any i, j with $1 \leq i < j \leq 4$, the central configuration satisfies

$$\frac{\partial U}{\partial r_{ij}^2} = -\lambda \frac{\partial S}{\partial r_{ij}^2} + \mu m' \frac{\partial I}{\partial r_{ij}^2}. \quad (10)$$

By (8) and (9), we have

$$\frac{\partial U}{\partial r_{ij}^2} = m_i m_j \varphi'(r_{ij}^2),$$

where $\varphi'(s)$ denotes the derivative of function $\varphi(s)$ with respect to s , and

$$m' \frac{\partial I}{\partial r_{ij}^2} = m_i m_j.$$

So, the equation (10) becomes

$$m_i m_j \varphi'(r_{ij}^2) = 32\lambda \Delta_i \Delta_j + \mu m_i m_j. \quad (11)$$

Using our assumption on masses, the equations of the central configurations become

$$\varphi'(r_{12}^2) = \nu \Delta_1 \Delta_2 + \mu, \quad (12)$$

$$\varphi'(r_{13}^2) = \frac{\nu}{\alpha} \Delta_1 \Delta_3 + \mu, \quad (13)$$

$$\varphi'(r_{14}^2) = \frac{\nu}{\alpha} \Delta_1 \Delta_4 + \mu, \quad (14)$$

$$\varphi'(r_{23}^2) = \frac{\nu}{\alpha} \Delta_2 \Delta_3 + \mu, \quad (15)$$

$$\varphi'(r_{24}^2) = \frac{\nu}{\alpha} \Delta_2 \Delta_4 + \mu, \quad (16)$$

$$\varphi'(r_{34}^2) = \frac{\nu}{\alpha^2} \Delta_3 \Delta_4 + \mu, \quad (17)$$

where $\nu = 32\lambda$.

According to Albouy[1][2], the geometrical relations between r_{ij}^2 and Δ_i are in the following

$$t_l = \sum_{i=1}^4 \Delta_i r_{il}^2, \quad t_1 = t_2 = t_3 = t_4. \quad (18)$$

Using the above implicit relations, Long and Sun, Perez-Chavela and Santoprete got the following Lemma:

Lemma 2.1. [8][10] *For a central configuration, the corresponding ν in the equations (12) – (17) is positive.*

3. The Proof of Theorem 1.1

Suppose $q = (q_1, q_2, q_3, q_4) \in (\mathbb{R}^2)^4$ is a planar central configuration as in the hypothesis of Theorem 1.1.

Our goal is to prove

$$\Delta_4 = -\Delta_3.$$

The way of proving Theorem 1.1 is by the contradiction argument. We assume that

$$\Delta_4 \neq -\Delta_3.$$

Lemma 3.1. *Under the hypothesis of Theorem 1.1, the following inequality holds :*

$$\Delta_3 < \Delta_1 < 0 < \Delta_2 < \Delta_4. \quad (19)$$

Proof. According to the paper [3], we know that $|\Delta_1| < |\Delta_3|$ and $|\Delta_2| < |\Delta_4|$. Because Δ_1, Δ_3 are negative and Δ_2, Δ_4 positive, we get $\Delta_3 < \Delta_1 < 0 < \Delta_2 < \Delta_4$. \square

Lemma 3.2. *If $\Delta_4 \neq -\Delta_3$, then*

$$b \neq e.$$

Proof. To prove the Lemma, we need to consider two possible cases:

Case 1. $\triangle_3 + \triangle_4 > 0$.

In this case, by the equation (6), we have

$$\begin{aligned}\triangle_1\triangle_3 - \triangle_2\triangle_4 &= \triangle_1\triangle_3 + (\triangle_1 + \triangle_3 + \triangle_4)\triangle_4 \\ &= \triangle_1\triangle_3 + \triangle_1\triangle_4 + \triangle_3\triangle_4 + \triangle_4^2 \\ &= (\triangle_1 + \triangle_4)(\triangle_3 + \triangle_4).\end{aligned}$$

We claim

$$\triangle_1 + \triangle_4 > 0.$$

In fact, if $\triangle_1 + \triangle_4 \leq 0$, then

$$\triangle_4 \leq -\triangle_1.$$

From the Lemma 3.1, we get

$$0 < \triangle_2 < \triangle_4 \leq -\triangle_1 < -\triangle_3.$$

It means

$$\triangle_2 + \triangle_3 < 0, \quad \triangle_1 + \triangle_4 \leq 0,$$

thus

$$\triangle_1 + \triangle_2 + \triangle_3 + \triangle_4 < 0,$$

which contradicts with the equation (6).

So, we have

$$\triangle_1\triangle_3 - \triangle_2\triangle_4 = (\triangle_1 + \triangle_4)(\triangle_3 + \triangle_4) > 0.$$

Thus we get

$$\triangle_1\triangle_3 > \triangle_2\triangle_4.$$

By $\alpha > 0$, we then obtain

$$\frac{\triangle_1\triangle_3}{\alpha} > \frac{\triangle_2\triangle_4}{\alpha}.$$

Since $\nu > 0$, thus by the equations (13), (16) and the monotonicity of $\varphi'(s)$, we have

$$b > e.$$

Case 2. $\triangle_3 + \triangle_4 < 0$.

In this case, the equation (6) implies that

$$\triangle_1\triangle_3 - \triangle_2\triangle_4 = (\triangle_1 + \triangle_4)(\triangle_3 + \triangle_4).$$

Since $\triangle_1 + \triangle_4 > 0$, we obtain

$$(\triangle_1 + \triangle_4)(\triangle_3 + \triangle_4) < 0.$$

Thus we get

$$\triangle_1\triangle_3 < \triangle_2\triangle_4.$$

By $\alpha > 0$, we then obtain

$$\frac{\triangle_1\triangle_3}{\alpha} < \frac{\triangle_2\triangle_4}{\alpha}.$$

Since $\nu > 0$, thus similar to Case 1, we obtain

$$b < e.$$

Therefore, in both cases, Lemma 3.2 holds. \square

From Lemma 3.2, we get

Lemma 3.3. *If $r_{13} = r_{24}$, then*

$$\triangle_4 = -\triangle_3.$$

Proof. We assume

$$\triangle_4 \neq -\triangle_3.$$

Using Lemma 3.2, we obtain

$$b \neq e.$$

It means $r_{13} \neq r_{24}$ and get a contradiction. So $\triangle_3 + \triangle_4 \neq 0$ is impossible. Finally, we get $\triangle_4 = -\triangle_3$. □

Lemma 3.4. *If $\triangle_4 = -\triangle_3$, then the quadrilateral q is a trapezoid.*

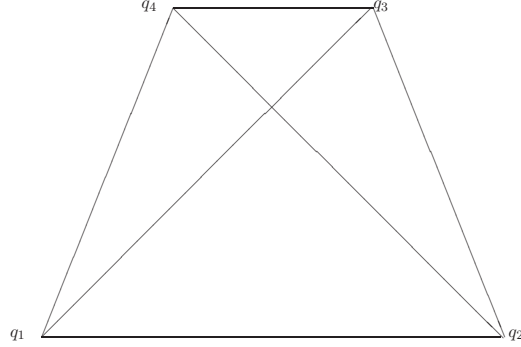


Figure.2

Proof. Since $\triangle_4 = -\triangle_3$, the areas of triangle $\triangle q_1 q_2 q_3$ and $\triangle q_1 q_2 q_4$ are equal. We also notice that the two triangles have a common edge r_{12} , thus their heights are equal. In other words, the quadrilateral q is a trapezoid. □

Now we can complete the proof of Theorem 1.1:

Proof of Theorem 1.1. Under the hypothesis of Theorem 1.1, we have $\triangle_4 = -\triangle_3$ from Lemma 3.2 and Lemma 3.3.

By the equation (6), we get $\triangle_2 = -\triangle_1$. It is clear that the configuration is a trapezoid.

By the equations (12)-(17) for the central configurations we have

$$\varphi'(r_{14}^2) = \frac{\nu}{\alpha} \triangle_1 \triangle_4 + \mu = \frac{\nu}{\alpha} \triangle_2 \triangle_3 + \mu = \varphi'(r_{23}^2), \quad (20)$$

Since $\varphi'(s)$ is an increasing function of s , we obtain

$$r_{14} = r_{23}.$$

Therefore, the configuration is an isosceles trapezoid. □

4. The Proof of Theorem 1.2

The way to prove Theorem 1.2 is similar to Theorem 1.1. Under the hypothesis of Theorem 1.2, Lemma 3.1 holds. Similar to Lemma 3.2, we have the following Lemma:

Lemma 4.1. *If $\Delta_4 \neq -\Delta_3$, then*

$$d \neq c.$$

Proof. To prove the lemma, we need to consider two possible cases:

Case 1. $\Delta_3 + \Delta_4 > 0$.

In this case, by the equation (5), we have

$$\Delta_2 + \Delta_4 > 0.$$

Using the equation (6), we obtain

$$\begin{aligned} \Delta_2\Delta_3 - \Delta_1\Delta_4 &= \Delta_2\Delta_3 + (\Delta_2 + \Delta_3 + \Delta_4)\Delta_4 \\ &= \Delta_2\Delta_3 + \Delta_2\Delta_4 + \Delta_3\Delta_4 + \Delta_4^2 \\ &= (\Delta_2 + \Delta_4)(\Delta_3 + \Delta_4) > 0. \end{aligned}$$

Thus we get

$$\Delta_2\Delta_3 > \Delta_1\Delta_4.$$

By $\alpha > 0$, we obtain

$$\frac{\Delta_2\Delta_3}{\alpha} > \frac{\Delta_1\Delta_4}{\alpha}.$$

Since $\nu > 0$, thus by equations (14), (15) and the monotonicity of $\varphi'(s)$, we have

$$d > c.$$

Case 2. $\Delta_3 + \Delta_4 < 0$.

In this case, the equation (5) implies that

$$\Delta_2 + \Delta_4 > 0.$$

By the equation (6), we have

$$\Delta_2\Delta_3 - \Delta_1\Delta_4 = (\Delta_2 + \Delta_4)(\Delta_3 + \Delta_4) < 0.$$

Thus we get

$$\Delta_2\Delta_3 < \Delta_1\Delta_4.$$

By $\alpha > 0$, we then obtain

$$\frac{\Delta_2\Delta_3}{\alpha} < \frac{\Delta_1\Delta_4}{\alpha}.$$

Since $\nu > 0$, thus similar to Case 1, we obtain

$$d < c.$$

Therefore, in both cases, Lemma 4.1 holds. □

Using the above Lemma, we get

Lemma 4.2. *If $r_{23} = r_{14}$, then*

$$\Delta_4 = -\Delta_3.$$

Proof. We assume

$$\Delta_4 \neq -\Delta_3.$$

Using Lemma 4.1, we obtain

$$d \neq c.$$

It means $r_{23} \neq r_{14}$ and get a contradiction. So $\Delta_3 + \Delta_4 \neq 0$ is impossible. Finally, we get $\Delta_4 = -\Delta_3$. □

Proof of Theorem 1.2. Under the hypothesis of Theorem 1.2, we have $\Delta_4 = -\Delta_3$ from Lemma 4.1 and Lemma 4.2.

By the equation (6), we get $\Delta_2 = -\Delta_1$. So the configuration q is a trapezoid. Since

$$r_{14} = r_{23},$$

we obtain that the configuration q is an isosceles trapezoid. □

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